Continued Finite Fractions and Euclid's Algorithm

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Abstract – A "general" continued fraction representation of a real number x is one of the form

$$x = a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cdots \cdots \cdots + \cfrac{b_n}{a_N}}}}$$

Where $a_0, a_1, \ldots$ and $b_1, b_2 \ldots$ are integers. In this article we define convergents of a finite continued fraction and continued fractions with positive quotients and discuss fraction algorithm and Euclid's algorithm.

Index Terms – Euclid Algorithm, Real number, Fraction.

1. INTRODUCTION

Define a function $f(\mathrm{n}) = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots \cdots \cdots + \cfrac{1}{a_N}}}}$ ……………….(a)

Consisting of N + 1 variables $a_0, a_1, \ldots \ldots, a_N$ as a finite continued fraction. As the representation (a) is cumbersome, we shall usually write it as $[a_0, a_1, \ldots \ldots, a_n]$ and we call $a_0, a_1, \ldots, a_n$ the partial quotients or simply the quotients of the finite continued fraction. As above we see that $[a_0] = \frac{a_0}{1}$, $[a_0, a_1] = \frac{a_0 a_1 + 1}{a_1}$, $[a_0, a_1, a_2] = \frac{a_2 a_1 a_0 + a_2 + a_0}{a_2 a_1 + 1}$ ……. Therefore $[a_0, a_1] = a_0 + \frac{1}{a_1}$ and

Similarly $[a_0, a_1, \ldots \ldots, a_{n-1}, a_n]$

$= [a_0, a_1, \ldots \ldots, a_{n-2}, a_{n-1} + \frac{1}{a_n}]$…………..(1.1)

i.e. $[a_0, a_1, \ldots \ldots, a_n] = a_0 + \frac{1}{[a_0, a_1, \ldots \ldots, a_n]}$

$= [a_0, [a_0, a_1, \ldots \ldots, a_n]]$, for $1 \leq n \leq$ N

Moreover $[a_0, a_1, \ldots \ldots, a_n] = [a_0, a_1, \ldots \ldots, a_{m-1}, [a_m, a_{m+1}, \ldots \ldots, a_n]]$

for $1 \leq n \leq$ N.

1.1. Definition: The quantity $[a_0, a_1, \ldots \ldots, a_n]$ for ($1 \leq n \leq$ N) is called nth convergent to $[a_0, a_1, \ldots \ldots, a_N]$. Also it is easy to find the convergents by means of the following theorem.

Theorem 1.2: Let p_n and q_n be defined as under $p_0 = a_0$, $p_1 = a_1 a_0 + 1$, $p_n = a_n\, p_{n-1} + p_{n-2}$ ($2 \leq n \leq$ N) and

$q_1 = 1$, $q_1 = a_1$, $q_n = a_n\, q_{n-1} + q_{n-2}$ ($2 \leq n \leq$ N) then $[a_0, a_1, \ldots \ldots, a_n] = \frac{p_n}{q_n}$.

Proof: For n=1 and n =1 theorem is obviously true.

Let suppose that result holds for n $\leq m$, where m < N. Then

$[a_0, a_1, \ldots \ldots, a_{m-1}, a_m] = \frac{p_m}{q_m} = \frac{a_m\, p_{m-1} + p_{m-2}}{a_m\, q_{m-1} + q_{m-2}}$, and p_{m-1}, p_{m-2}, q_{m-1}, q_{m-2} depend only upon $a_0, a_1, \ldots \ldots, a_{m-1}$.

Hence using (1.1) we get $[a_0, a_1, \ldots \ldots, a_{m-1}, a_m, a_{m+1}]$

$= [a_0, a_1, \ldots \ldots, a_{m-1}, a_m + \frac{1}{a_{m+1}}]$

$$= \frac{\left(a_m + \frac{1}{a_{m+1}}\right) p_{m-1} + p_{m-2}}{\left(a_m + \frac{1}{a_{m+1}}\right) q_{m-1} + q_{m-2}} = \frac{a_{m+1}(a_m p_{m-1} + p_{m-2}) + p_{m-1}}{a_{m+1}(a_m q_{m-1} + q_{m-2}) + q_{m-1}} = \frac{a_{m+1} p_m + p_{m-1}}{a_{m+1} q_m + q_{m-1}} = \frac{p_{m+1}}{q_{m+1}}$$

Hence by induction the theorem is proved.

Note: From $p_0 = a_0$, $p_1 = a_1 a_0 + 1$, $p_n = a_n\, p_{n-1} + p_{n-2}$ ($2 \leq n \leq$ N) and

$q_1 = 1$, $q_1 = a_1$, $q_n = a_n\, q_{n-1} + q_{n-2}$ ($2 \leq n \leq$ N) it follows that

$\frac{p_n}{q_n} = \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}}$

Also $p_n q_{n-1} - p_{n-1} q_n = (a_n\, p_{n-1} + p_{n-2})\, q_{n-1}$

$- p_{n-1}(a_n\, q_{n-1} + q_{n-2})$

$= -(p_{n-1} q_{n-2} - p_{n-2} q_{n-1})$.

Repeating the argument with n-1, n-2,……,2 in place of n, we get

$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1} (p_1 q_0 - p_0 q_1) = (-1)^{n-1}$.

Also $p_n q_{n-2} - p_{n-2} q_n = (a_n\, p_{n-1} + p_{n-2})\, q_{n-2} - p_{n-2}(a_n\, q_{n-1} + q_{n-2})$

$= a_n (p_{n-1} q_{n-2} - p_{n-2} q_{n-1}) = (-1)^{n-1} a_n$.

Remark: The functions p_n and q_n satisfies the following.

$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$ or $\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^{n-1}}{q_{n-1}\, q_n}$

Also they satisfy $p_n q_{n-2}$- $p_{n-2} q_n$ = $(-1)^{n-1} a_n$ or $\frac{p_n}{q_n}$ - $\frac{p_{n-2}}{q_{n-2}}$ = $\frac{(-1)^{n-1} a_n}{q_{n-2}\, q_n}$.

1.3.Definition: Now we assign numerical values to the quotients a_n so to the fraction $a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots \ldots\ldots + \frac{1}{a_N}}}}$ and to its convergents.

Now suppose that $a_1 > 0$,…….., $a_N > 0$, a_0 may be negative , in this case the continued fraction is said to be simple. Write $x_n = \frac{p_n}{q_n}$, x = x_N so that the value of the continued fraction is x_N or x. Then

$[a_0, a_1, \ldots \ldots, a_N]$ = $[a_0, a_1, \ldots \ldots, a_{n-1,} [a_n, a_{n+1}, \ldots \ldots, a_N]]$

= $\frac{[a_n, a_{n+1}, \ldots\ldots, a_N] p_{n-1} + p_{n-2}}{[a_n, a_{n+1}, \ldots\ldots, a_N] q_{n-1} + q_{n-2}}$ for 2≤ n ≤ N.

Note: As every q_n is positive then from $\frac{p_n}{q_n}$ - $\frac{p_{n-2}}{q_{n-2}}$ = $\frac{(-1)^{n-1} a_n}{q_{n-2}\, q_n}$ and $a_1 > 0$,…….., $a_N > 0$, $x_n - x_{n-2}$ has the sign of $(-1)^n$. Which proves that the even convergents x_{2n} increase strictly with n, while the odd convergents x_{2n+1} decrease strictly.

Also from $\frac{p_n}{q_n}$ - $\frac{p_{n-1}}{q_{n-1}}$ = $\frac{(-1)^{n-1}}{q_{n-1}\, q_n}$, $x_n - x_{n-1}$ has the sign of $(-1)^{n-1}$

so that $x_{2m+1} > x_{2m}$ contrary if we assume that $x_{2m+1} \le x_{2\mu}$ for some m, μ.If m < μ then from above $x_{2m+1} < x_{2m}$, and if m < μ

then $x_{2\mu+1} < x_{2\mu}$which is a contradiction. Hence we say that every odd convergent is greater than any even convergent.

1.4.Definition: If all a_n are integers then the continued fraction is called Simple Fraction. If p_n and q_n are integers and q_n is positive then

$[a_0, a_1, \ldots \ldots, a_N]$ = $\frac{p_N}{q_N}$ = x, we say that the number x (which is necessarily rational) is represented by the continued fraction.

Theorem 1.5: $q_n \ge n$, with inequality when n > 3.

Proof: In the first place, q_0 = 1, q_1 = a_1 ≥ 1. If n ≥ 2 then

$q_n = a_n q_{n-1} + q_{n-2} \ge q_{n-1}$+1 so that $q_n > q_{n-1}$ and $q_n \ge n$. If n > 3, then

$q_n \ge q_{n-1} + q_{n-2} > q_{n-1}$+1 ≥ n, and so $q_n > n$.

1.6.Definition: Any simple continued fraction $[a_0, a_1, \ldots \ldots, a_N]$ represents a rational number x = x_N

Theorem 1.7: If x is representable by a simple continued fraction with an odd (even) number of convergents, it is also representable by one with an even (odd) number.

Proof: If $a_n \ge 2$ then $[a_0, a_1, \ldots \ldots, a_n]$

= $[a_0, a_1, \ldots \ldots, a_n - 1, 1]$ while, if a_n = 1,

$[a_0, a_1, \ldots \ldots, a_{n-1}, 1]$ = $[a_0, a_1, \ldots \ldots, a_{n-2}, a_n + 1]$

For example [2,2,3] = [2,2,2,1] this choice of alternative representations is often useful. We call a'_n = $[a_n, a_{n+1}, \ldots \ldots, a_N]$ (0 ≤ n ≤ N) the nth complete quotient of the continued fraction $[a_0, a_1, \ldots \ldots, a_N]$. Thus x = a'_0, x = $\frac{a'_1\, a_0 + 1}{a'_1}$ and

x = $\frac{a'_n p_{n-1} + p_{n-2}}{a'_n q_{n-1} + q_{n-2}}$,

(2 ≤ n ≤ N) …………….(b)

Theorem 1.8: a_n = $[a'_n]$, the integral part of a'_n except that a_{N-1} = $[a_{N-1}] - 1$ when a_N = 1.

Proof: If N = 0, then $a_0 = a'_0$= $[a'_0]$. If N > 0 then a'_n= a_n+ $\frac{1}{a'_{n+1}}$ (0 ≤ n ≤ N-1).

Now a'_{n+1} > 1 (0 ≤ n ≤ N-1) except that a'_{n+1} = 1 when n = N -1 and a_N = 1.

Hence $a_n < a'_n < a_n + 1$ (0 ≤ n ≤ N-1) and a_n = $[a'_n]$ for (0 ≤ n ≤ N-1) except in the case specified. And in any case a_N = $a'_N = [a'_N]$.

Theorem 1.9: If two simple continued fractions $[a_0, a_1, \ldots \ldots, a_N]$ and $[b_0, b_1, \ldots \ldots, b_M]$ have the same value x, and $b_M > 1$, then M = N and the fractions are identical.

Proof: When we say that the two continued fractions are identical we mean that they are formed by the same sequence of partial quotients.

By the above theorem a_0 = [x] = b_0. Let us suppose that the first n partial quotients in the continued fractions are identical and that a'_n *and* b'_n are the nth complete quotients. Then x = $[a_0, a_1, \ldots \ldots, a_{n-1}, a'_n]$ = $[a_0, a_1, \ldots \ldots, a_{n-1}, b'_n]$.

If n = 1 then $a_0 + \frac{1}{a'_1} = a_0 + \frac{1}{b'_1}$, $a'_1 = b'_1$, and therefore by above theorem $a_1 = b_1$.

If n > 1, then by $\frac{a'_n p_{n-1} + p_{n-2}}{a'_n q_{n-1} + q_{n-2}} = \frac{b'_n p_{n-1} + p_{n-2}}{b'_n q_{n-1} + q_{n-2}}$,

$(a'_n - b'_n)(p_{n-1} q_{n-2} - p_{n-2} q_{n-1})$ = 0. But $p_{n-1} q_{n-2} - p_{n-2} q_{n-1}$ = $(-1)^n$ then

as $p_n q_{n-1} - p_{n-1} q_n$ = $(-1)^{n-1}$ and so $a'_n = b'_n$,

it follows from the above theorem that $a_n = b_n$.

Suppose now for example, that N ≤ M. Then our argument shows that $a_n = b_n$ for N ≤ M. If M > N then $\frac{p_N}{q_N}$ = $[a_0, a_1, \ldots \ldots, a_N]$ = $[a_0, a_1, \ldots \ldots, a_N, b_{N+1}, \ldots \ldots, b_M]$

$= \frac{b'_{N+1}p_N + p_{N-1}}{b'_{N+1}q_N + q_{N-1}}$, Hence by (b) $p_N q_{N-1} - p_{N-1}q_N = 0$ which is false. Hence M = N and the fractions are identical.

2. CONTINUED FRACTION ALGORITHM AND EUCLID'S ALGORITHM

Let x be any real number, and let a_0 = [x]. Then x = $a_0 + \xi_0$, $0 \le \xi_0 < 1$.

If $\xi_0 \neq 0$, we can write $\frac{1}{\xi_0} = a'_1$, $[a'_n] = a_1$, a'_1

$= a_1 + \xi_1$, $0 \le \xi_1 < 1$.

If $\xi_1 \neq 0$, we can write $\frac{1}{\xi_1} = a'_2 = a_2 + \xi_2$,

$0 \le \xi_2 < 1$, and so on

Also $a'_n = \frac{1}{\xi_{n-1}} > 1$, and so $a_n \ge 1$, for $n \ge 1$.

Thus x = $[a_0, a'_1] = [a_0, a_1 + \frac{1}{a'_2}] = [a_0, a_1, a'_2] = [a_0, a_1, a_2, a'_3]$ = …….. where $a_0, a_1, a_2, \ldots$ are integers and

$a_1 > 0$, $a_2 > 0$, ……….

The system of equations x = $a_0 + \xi_0$, $(0 \le \xi_0 < 1)$,

$\frac{1}{\xi_0} = a'_1 = a_1 + \xi_1$, $(0 \le \xi_1 < 1)$,

$\frac{1}{\xi_1} = a'_2 = a_2 + \xi_2$, $(0 \le \xi_2 < 1)$,

………….. is known as the continued fraction algorithm. The algorithm continues so long as $\xi_n \neq 0$. If we eventually reach a value of n, say N, for which $\xi_N = 0$, the algorithm terminates and x = $[a_0, a_1, \ldots, a_N]$.

In this case x is represented by a simple continued fraction, and is rational. The number a'_n are the complete quotients of the continued fraction.

Theorem 2.1: Any rational number can be represented by a finite simple continued fraction.

Proof: If x is an integer, then $\xi_0 = 0$ and x = a_0. If x is not integral, then x = $\frac{h}{k}$, where h and k are integers and k > 1. Since $\frac{h}{k} = a_0 + \xi_0$, h = a_0k + ξ_0k, a_0 is the quotient, and $k_1 = \xi_0$k the remainder, when h is divided by k.

If $\xi_0 \neq 0$ then $a'_1 = \frac{1}{\xi_0} = \frac{k}{k_1}$ and $\frac{k}{k_1} = a_1 + \xi_1$,

k = $a_1 k_1 + \xi_1 k_1$; thus a_1 is the quotient, and

$k_2 = \xi_1 k_1$ the remainder, when k is divided by k_1. Thus we obtain a series of equations h = a_0k + k_1 ,

k = $a_1 k_1 + k_2$, $k_1 = a_2 k_2 + k_3$,…………….

Continuing so long as $\xi_n \neq 0$, or what is the same thing, so long as $k_{n+1} \neq 0$.

The non-negative integers k, $k_1, k_2, \ldots$ form a strictly decreasing sequence, and so $k_{n+1} = 0$ for

some N. It follows that $\xi_N = 0$ for some N, and the continued fraction algorithm terminates. This proves the theorem.

Remark: The system of equations

h = a_0k + k_1 , $(0 < k_1 < k)$,

k = $a_1 k_1 + k_2$, $(0 < k_2 < k_1)$,

………………………………..

$k_{N-2} = a_{N-1}k_{N-1} + k_N$, $(0 < k_N < k_{N-1})$,

$k_{N-1} = a_N k_N$ is known as Euclid's algorithm.

3. DIFFERENCE BETWEEN THE FRACTION AND ITS CONVERGENTS:

Suppose N > 1 and n > 0 then by x = $\frac{a'_n p_{n-1} + p_{n-2}}{a'_n q_{n-1} + q_{n-2}}$, $(1 \le n \le N-1)$ and so

x - $\frac{p_n}{q_n} = -\frac{p_n q_{n-1} - p_{n-1} q_n}{q_n (a'_{n+1} q_n + q_{n-1})} = \frac{(-1)^n}{q_n (a'_{n+1} q_n + q_{n-1})}$, Also x - $\frac{p_0}{q_0}$ = x - $a_0 = \frac{1}{a'_1}$.

If we write $q'_1 = a'_1$, $q'_n = a'_n q_{n-1} + q_{n-2}$, $(1 \le n \le N-1)$

(So in particular $q'_N = q_N$), we have the following theorem.

Theorem 3.1: If $1 \le n \le N-1$, then

$$x - \frac{p_n}{q_n} = \frac{(-1)^n}{q_n\, q'_{n+1}}$$

Proof: $a_{n+1} < a'_{n+1} < a_{n+1} + 1$ for $n \le N-2$,

by the equation $a_n < a'_n < a_n + 1$ $(0 \le n \le N-1)$, except that $a'_{N-1} = a_{N-1} + 1$

when a_N =1. Hence if we ignore this exceptional case for the moment, we have

$q_1 = a_1 < a'_1 + 1 \le q_2$ and $q'_{n+1} = a'_{n+1} q_n + q_{n-1}$

$> a_{n+1} q_n + q_{n-1} = q_{n+1}$

$q'_{n+1} < a_{n+1} q_n + q_{n-1} + q_n = q_{n+1} + q_n$

$\le a_{n+2} q_{n+1} + q_n = q_{n+2}$,

for $1 \le n \le N-2$. It follows that $\frac{1}{q_{n+2}} < |p_n - q_n x|$

$< \frac{1}{q_{n+1}}$ $(n \le N-2)$ while $|p_{N-1} - q_{N-1}x| = \frac{1}{q_N}$, $p_N - q_N x = 0$ in the exceptional case

$q'_{n+1} < a_{n+1} q_n + q_{n-1} + q_n = q_{n+1} + q_n$

$\leq a_{n+2}q_{n+1} + q_n$ = q_{n+2} must be replaced by

q'_{N-1} = (|a_{N-1} + 1) q_{N-2}+ q_{N-3} = q_{N-1}+ q_{N-2} = q_N and the first inequality. In the case $\frac{1}{q_{n+2}}$ < $|p_n - q_n$x| < $\frac{1}{q_{n+1}}$ ($n \leq$ N-2) by an equality. In this case shows that $|p_n - q_n$x| decreases steadily as n increases, Since q_n increases steadily, |x - $\frac{p_n}{q_n}$| decreases steadily.

We may sum up the most important conclusion in the following theorem

i.e. If N >1, n >0 then the differences x - $\frac{p_n}{q_n}$, q_nx - p_n = $\frac{(-1)^n\delta_n}{q_{n+1}}$, where 0 < δ_n < 1 (1 ≤ n ≤ N-2),

δ_{N-1} = 1, |x - $\frac{p_n}{q_n}$| ≤ $\frac{1}{q_n q_{n+1}}$ < $\frac{1}{q_n^2}$ for n ≤ N-1 with inequality in both places except when n = N – 1.

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