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A Ternary Quadratic Diophantine Equation $x^2+y^2=10z^2$

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Abstract – The Quadratic Diophatine equation with three unknowns represented by $x^2+y^2=10z^2$ is analyzed for finding its non-zero distinct integral solutions. Different patterns of solutions of the equation under consideration are obtained. A few interesting properties among the solutions are presented.

Index Terms – Ternary quadratic equation with three unknowns, Integral solutions, Polygonal numbers, Pyramidal numbers and Special numbers.

2010 Mathematics Subject Classification: 11D09.

1. INTRODUCTION

1.1. Notations

$t_{m,n}$ = Polygonal number of rank n with sides m

p_m^n = Pyramidal number of rank n with sides m

$ct_{m,n}$ = Centered Polygonal number of rank n with sides m

p_n = Pronic number

g_n = Gnomonic number

Tha_n = Thabit-ibn-Kurrah number

$car1_n$ = Carol number

Mer_n = Mersenne number

ky_n = Kynea number

The ternary homogeneous quadratic Diophantine equation offers an unlimited field for research because of their variety [1-2]. For an extensive review of various problems one may refer [3-11]. In this context one may also see [12-23]. This communication concerns with yet another interesting ternary quadratic equation $x^2+y^2=10z^2$ for determining its infinitely many non-zero integral solutions. Also a few interesting properties among the solutions are presented.

2. METHOD OF ANALYSIS

The ternary quadratic equation representing homogeneous cone is

$$x^2+y^2=10z^2 \tag{1}$$

We present below different patterns of non-zero distinct integral solutions to (1)

2.1. Pattern:1

The substitution of linear transformations $(u \neq \sigma \neq 0)$

$$x=u+\sigma,\ y=u-\sigma,\ z=\alpha \tag{2}$$

in (1) leads to

$$u^2+\sigma^2=5\alpha^2 \tag{3}$$

whose initial solution is

$\alpha_0=\sigma$ and $u_0=2\sigma$

Let $(\tilde{\alpha}_n, \tilde{u}_n)$ be the general solution of the Pellian

$$u^2=5\alpha^2+1$$

Where

$$\tilde{\alpha}_n=\frac{1}{2\sqrt{5}}\left[\left(9+4\sqrt{5}\right)^{n+1}-\left(9-4\sqrt{5}\right)^{n+1}\right]$$

$$\tilde{u}_n=\frac{1}{2}\left[\left(9+4\sqrt{5}\right)^{n+1}+\left(9-4\sqrt{5}\right)^{n+1}\right],$$

$n=0,1,2,\ldots$

Applying Brahmagupta's lemma between the solutions (α_0, u_0) and $(\tilde{\alpha}_n, \tilde{u}_n)$ the sequence of values of α and u satisfying equation (3) is given by

$$\alpha_{n+1} = \frac{\sigma}{2}\left[\left(9+4\sqrt{5}\right)^{n+1} + \left(9-4\sqrt{5}\right)^{n+1}\right] + \frac{\sigma}{\sqrt{5}}\left[\left(9+4\sqrt{5}\right)^{n+1} - \left(9-4\sqrt{5}\right)^{n+1}\right] \quad (4$$

$$u_{n+1} = \sigma\left[\left(9+4\sqrt{5}\right)^{n+1} + \left(9-4\sqrt{5}\right)^{n+1}\right] + \frac{\sigma\sqrt{5}}{2}\left[\left(9+4\sqrt{5}\right)^{n+1} - \left(9-4\sqrt{5}\right)^{n+1}\right] \quad (5)$$

Substituting α_{n+1} and u_{n+1} in equation (2), we get the values of x_{n+1}, y_{n+1} and z_{n+1}. The non-zero distinct integrals values of x_{n+1}, y_{n+1} and z_{n+1} satisfying (1) are given by

$$x_{n+1} = \sigma\left(\left(\left(9+4\sqrt{5}\right)^{n+1} + \left(9-4\sqrt{5}\right)^{n+1}\right) + \frac{\sqrt{5}}{2}\left(\left(9+4\sqrt{5}\right)^{n+1} - \left(9-4\sqrt{5}\right)^{n+1}\right) + 1\right)$$

$$y_{n+1} = \sigma\left(\left(\left(9+4\sqrt{5}\right)^{n+1} + \left(9-4\sqrt{5}\right)^{n+1}\right) + \frac{\sqrt{5}}{2}\left(\left(9+4\sqrt{5}\right)^{n+1} - \left(9-4\sqrt{5}\right)^{n+1}\right) - 1\right)$$

$$z_{n+1} = \sigma\left(\frac{1}{2}\left(\left(9+4\sqrt{5}\right)^{n+1} + \left(9-4\sqrt{5}\right)^{n+1}\right) + \frac{1}{\sqrt{5}}\left(\left(9+4\sqrt{5}\right)^{n+1} - \left(9-4\sqrt{5}\right)^{n+1}\right)\right)$$

The recurrence relations on x_{n+1}, y_{n+1} and z_{n+1} are found to be

$$x_{n+3} - 18x_{n+2} + x_{n+1} = -16\sigma$$

$$y_{n+3} - 18y_{n+2} + y_{n+1} = 16\sigma$$

$$z_{n+3} - 18z_{n+2} + z_{n+1} = 0$$

2.2. Pattern:2

Write (1) as

$$10z^2 - y^2 = x^2 * 1 \quad (6)$$

Write 1 as

$$1 = \left(\sqrt{10}+3\right)\left(\sqrt{10}-3\right) \quad (7)$$

Assume $x = 10a^2 - b^2$ (8)

where $a, b \succ 0$

Using (7) and (8) in (6), we get

$$\left(\sqrt{10}z + y\right)\left(\sqrt{10}z - y\right) = \left(\sqrt{10}a + b\right)^2\left(\sqrt{10}a - b\right)^2\left(\sqrt{10}+3\right)\left(\sqrt{10}-3\right)$$

On employing the method of factorization and equating the rational and irrational parts, we get

$$z = z(a,b) = 10a^2 + b^2 + 6ab \quad (9)$$

$$y = y(a,b) = 30a^2 + 3b^2 + 20ab$$

Thus (8) and (9) represents non-zero distinct integral solutions of (1).

Properties:

$$x\left(2^n,1\right) + z\left(2^n,1\right) = 18Mer_{2n} + ky_n + car\,1_n + 2Tha_n + 22$$

$$y(a,1) - z(a,1) - t_{42,a} \equiv 2 \pmod{33}$$

2.3. Pattern:3

Instead of (7), write 1 as

$$1 = \frac{\left(\sqrt{10}+1\right)\left(\sqrt{10}-1\right)}{9} \quad (10)$$

Using (10) and (8) in (6) and following the procedure as in pattern (2) the non-zero distinct solutions of (1) are given by

$$z = \frac{1}{3}\left(10a^2 + b^2 + 2ab\right)$$

$$y = \frac{1}{3}\left(10a^2 + b^2 + 20ab\right)$$

Choosing $a = 3A,\ b = 3B$ the corresponding integral values of x, y and z satisfying the (1) are given by

$$x = x(A,B) = 90A^2 - 9B^2$$

$$y = y(A,B) = 30A^2 + 3B^2 + 60AB$$

$$z = z(A,B) = 30A^2 + 3B^2 + 6AB$$

Properties:

1. $x\left(2^n, 2n+1\right) = 30Tha_{2n} - 36p_n + 21$
2. $y(n+1,n+1) - z(n+1,n+1) = 2\left(t_{56,n} + 40g_n + 67\right)$

2.4. Pattern: 4

Write (1) as

$$9z^2+z^2=x^2+y^2$$

$$9z^2-y^2=x^2-z^2$$

$$(3z+y)(3z-y)=(x+z)(x-z) \qquad (11)$$

Write (11) as

$$\frac{3z+y}{x-z}=\frac{x+z}{3z-y}=\frac{A}{B},\ B\neq 0$$

This is equivalent to the following two equations

$$-xA+yB+z(A+3B)=0$$

$$xB+yA+z(B-3A)=0$$

On employing the method of cross multiplication, we get

$$\left.\begin{aligned} x&=-A^2+B^2-6AB \\ y&=-3A^2+3B^2+2AB \\ z&=-A^2-B^2 \end{aligned}\right\} \qquad (12)$$

Thus (12) represents non-zero distinct integral solutions of (1).

Properties:

1. $x(A+1,1)+z(A+1,1)+t_{12,A}-t_{8,A}+6g_A+14=0$

2. $y(2,B^2+1)+3=ct_{6,B^2}+t_{16,B}+3g_B$

2.5. Pattern:5

Write 10 as

$$10=(3+i)(3-i) \qquad (13)$$

Assume, $z=a^2+b^2$ (14)

Using (13) and (14) in (1),we get

$$(x+iy)(x-iy)=(3+i)(3-i)(a+ib)^2(a-ib)^2$$

On employing the method of factorization and equating the real and imaginary parts, we get

$$x=x(a,b)=3(a^2-b^2)-2ab \qquad (15)$$

$$y=y(a,b)=a^2-b^2+6ab$$

Thus (14) and (15) represents non-zero distinct integral solutions of (1).

Properties:

1. $z(n,2n+1)-3x(n+1,n)+9g_n+18+n=ct_{20,n}+2t_{3,n}$

2. $y(n+1,n)-ct_{12,n}-2g_n=0$

Generation of solution:

Let (x_0,y_0,z_0) be the initial solution of (1). Then, each of the following triple of nonzero distinct integers based on x_0,y_0 and z_0 also satisfies (1)

Triple 1: $\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix}=\begin{pmatrix} \frac{Y_{n+1}+1}{2} & \frac{Y_{n+1}-1}{2} & 10X_{n+1} \\ \frac{Y_{n+1}-1}{2} & \frac{Y_{n+1}+1}{2} & 10X_{n+1} \\ X_{n+1} & X_{n+1} & Y_{n+1} \end{pmatrix}$

Where

$$X_{n+1}=\frac{1}{2\sqrt{20}}\left[\left(9+2\sqrt{20}\right)^{n+1}-\left(9-2\sqrt{20}\right)^{n+1}\right]$$

$$Y_{n+1}=\frac{1}{2}\left[\left(9+2\sqrt{20}\right)^{n+1}+\left(9-2\sqrt{20}\right)^{n+1}\right]$$

3. CONCLUSION

One may search for other patterns of solution and their corresponding properties.

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