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On The Binary Quadratic Diophantine Equation $x^2 - 6xy + y^2 + 24x = 0$

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Abstract – The binary quadratic equation $x^2 - 6xy + y^2 + 24x = 0$ represents a hyperbola. In this paper we obtain a sequence of its integral solutions and present a few interesting relations among them.

Index Terms – Binary quadratic equation, integral solutions, MSC SUBJECT CLASSIFICATION: 11D09.

1. INTRODUCTION

The binary quadratic Diophantine equations (both homogeneous and non-homogeneous) are rich in variety [1-6].In [7-16] the binary quadratic non-homogeneous equations representing hyperbolas respectively are studied for their non-zero integral solutions. These results have motivated us to search for infinitely many non-zero integral solutions of another interesting binary quadratic equation given by $x^2 - 6xy + y^2 + 24x = 0$. The recurrence relations satisfied by the solutions x and y are given. Also a few interesting properties among the solutions are exhibited.

2. METHOD OF ANALYSIS

The Diophantine equation representing the binary quadratic equation to be solved for its non-zero distinct integral solution is

$$x^2 - 6xy + y^2 + 24x = 0 \qquad (1)$$

Note that (1) is satisfied by the following non-zero integer pairs

$$(-3,3),(3,9),(6,6),(6,30),(27,9),\ (150,30)$$

However, we have other solutions for (1), which are illustrated below:

Solving (1) for y, we have

$$y = 3x \pm \sqrt{8x^2 - 24x} \qquad (2)$$

Let $\alpha^2 = 8x^2 - 24x$

Multiplying the above equation by 8 on both sides and performing a few calculations, we have

$$X^2 = 8\alpha^2 + 144 \qquad (3)$$

where $X = 8x - 12$ (4)

The least positive integer solution of (3) is

$$\alpha_0 = 12, X_0 = 36$$

Now,to find the other solution of (3),consider the pellian equation

$$X^2 = 8\alpha^2 + 1 \qquad (5)$$

Whose fundamental solution is

$$\left(\tilde{\alpha}_0, \tilde{X}_0\right) = (1,3)$$

The other solutions of (5) can be derived from the relations

$$\tilde{X}_n = \frac{f_n}{2}, \tilde{\alpha}_n = \frac{g_n}{2\sqrt{8}}$$

Where

$$f_n = \left(3+\sqrt{8}\right)^{n+1} + \left(3-\sqrt{8}\right)^{n+1}$$
$$g_n = \left(3+\sqrt{8}\right)^{n+1} - \left(3-\sqrt{8}\right)^{n+1}$$

Applying the lemma of Brahmagupta between (α_0, X_0) & $\left(\tilde{\alpha}_n, \tilde{X}_n\right)$, the other solutions of (3) can be obtained from the relation

$$\alpha_{n+1} = 6\ f_n + \frac{18}{\sqrt{8}}\ g_n \quad (6)$$

$$X_{n+1} = 18\ f_n + \frac{48}{\sqrt{8}}\ g_n \quad (7)$$

Taking positive sign on the R.H.S of (2) and using (4),(6)&(7), the non-zero distinct integer solutions of the hyperbola (1) are obtained as follows

$$x_{n+1} = \frac{1}{8}\left(18 f_n + \frac{48}{\sqrt{8}} g_n + 12\right) \quad (8)$$

$$y_{n+1} = 3x_{n+1} + \frac{1}{8}\left(102 f_n + 36\sqrt{8} g_n + 36\right), n = -1,0,1,2,3... \quad (9)$$

The recurrence relations for x_{n+1}, y_{n+1} are respectively

$$8x_{n+1} - 48x_{n+2} + 8x_{n+3} = -48$$
$$48y_{n+2} - 8y_{n+1} - 8y_{n+3} = 144$$

A few numerical examples are given in table below

Table:NUMERICAL SOLUTIONS

n	x_{n+1}	y_{n+1}
-1	6	30
0	27	153
1	150	870
2	867	5049

Some relations satisfied by the solutions (8) & (9) are as follows

1) $x_{n+2} = y_{n+1} - 3$
2) $x_{n+3} = 6y_{n+1} - x_{n+1} - 24$
3) $y_{n+2} = 6y_{n+1} - x_{n+1} - 21$
4) $y_{n+3} = 35y_{n+1} - 6x_{n+1} - 144$
5) $x_{n+1} = 6x_{n+2} - y_{n+2} - 3$
6) $x_{n+3} = y_{n+2} - 3$
7) $y_{n+3} = -x_{n+2} + 6y_{n+2} - 21$
8) $x_{n+1} = 35x_{n+3} - 6y_{n+3} - 24$
9) $x_{n+2} = 6x_{n+3} - y_{n+3} - 3$
10) $y_{n+1} = 6x_{n+3} - y_{n+3}$
11) $y_{n+2} = x_{n+3} + 3$
12) Each of the following expressions is a nasty number

i) $48x_{2n+2} - 8y_{2n+2} - 24$

ii) $280x_{2n+2} - 48y_{2n+2} - 192$

13)
$$\frac{1}{6}\left[48x_{3n+3} - 8y_{3n+3} - 36\right] + 3\left[\frac{1}{6}\left(48x_{n+1} - 8y_{n+1} - 36\right)\right]$$
is a cubic number.

2.1. Remarkable Observations

1) By considering suitable linear transformations between the solutions of (1), one may get integer solutions for hyperbolas

Example 1) Define $X = 48x_{n+1} - 8y_{n+1} - 36$, $Y = -136x_{n+1} + 24y_{n+1} + 96$

Note that the pair (X, Y) satisfies the hyperbola $Y^2 = 8X^2 - 32 \times 6^2$

Example 2) Define
$X = 1680x_{n+2} - 288y_{n+2} - 1224$,
$Y = -4752x_{n+2} + 816y_{n+2} + 3456$

Note that the pair (X, Y) satisfies the hyperbola $Y^2 = 8X^2 - 32 * 36^2$

2) By considering suitable linear transformations between the solutions of (1) , one may get integer solutions for parabolas

Example 3) Define

$X = 48x_{n+1} - 8y_{n+1} - 36$, $Y = -136x_{n+1} + 24y_{n+1} + 96$

Note that the pair (X, Y) satisfies the parabola $Y^2 = 6 * 8X - 32 * 6^2$

Example 4) Define
$X = 1680x_{n+2} - 288y_{n+2} - 1224$,
$Y = -4752x_{n+2} + 816y_{n+2} + 3456$

Note that the pair (X, Y) satisfies the parabola $Y^2 = (36 * 8)X - 32 * (36^2)$

Solving (1) for x, we have

$$x = 3y - 12 \pm \sqrt{8y^2 + 144 - 72y} \quad (10)$$

Let $\alpha^2 = 8y^2 + 144 - 72y$

Multiplying the above equation by 8 on both sides and performing a few calculations, we have

$$Y^2 = 8\alpha^2 + 144 \quad (11)$$

where $Y = 8y - 36$ (12)

The least positive integer solution of (3) is

$$\alpha_0 = 12, Y_0 = 36$$

Now, to find the other solution of (11), consider the pellian equation

$$Y^2 = 8\alpha^2 + 1 \quad (13)$$

whose fundamental solution is

$$\left(\tilde{\alpha}_0, \tilde{Y}_0\right) = (1,3)$$

The other solutions of (13) can be derived from the relations

$$\tilde{Y}_n = \frac{f_n}{2}, \tilde{\alpha}_n = \frac{g_n}{2\sqrt{8}}$$

where

$$f_n = \left(3+\sqrt{8}\right)^{n+1} + \left(3-\sqrt{8}\right)^{n+1}$$
$$g_n = \left(3+\sqrt{8}\right)^{n+1} - \left(3-\sqrt{8}\right)^{n+1}$$

Applying the lemma of Brahmagupta between $\left(\alpha_0, Y_0\right) \& \left(\tilde{\alpha}_n, \tilde{Y}_n\right)$, the other solutions of (11) can be obtained from the relation

$$\alpha_{n+1} = 6 f_n + \frac{18}{\sqrt{8}} g_n \quad (14)$$

$$Y_{n+1} = 18 f_n + \frac{48}{\sqrt{8}} g_n \quad (15)$$

Taking positive sign on the R.H.S of (10) and using (12),(14)&(15), the non-zero distinct integer solutions of the hyperbola (1) are obtained as follows

$$y_{n+1} = \frac{1}{8}\left(18 f_n + \frac{48}{\sqrt{8}} g_n + 36\right) \quad (16)$$

$$x_{n+1} = 3y_{n+1} + \frac{1}{8}\left(102 f_n + 36\sqrt{8} g_n + 12\right), n = -1,0,1,2,3... \quad (17)$$

The recurrence relations for x_{n+1}, y_{n+1} are respectively

$$48x_{n+2} - 8x_{n+1} - 8x_{n+3} = 48$$
$$8y_{n+1} - 48y_{n+2} + 8y_{n+3} = -144$$

A few numerical examples are given in table below

Table: NUMERICAL SOLUTIONS

n	x_{n+1}	y_{n+1}
-1	27	9
0	150	30
1	867	153
2	5046	870

Some relations satisfied by the solutions (16) & (17) are as follows

1) $8y_{n+2} = 8x_{n+1} + 24$
2) $8y_{n+3} = -8y_{n+1} + 48x_{n+1}$
3) $8x_{n+2} = -8y_{n+1} + 48x_{n+1} - 24$
4) $8x_{n+3} = -48y_{n+1} + 280x_{n+1} - 192$
5) $8y_{n+1} = 48y_{n+2} - 8x_{n+2} - 168$
6) $8y_{n+3} = 8x_{n+2} + 24$
7) $8x_{n+1} = 8y_{n+2} - 24$
8) $8x_{n+3} = -8y_{n+2} + 48x_{n+2} - 24$
9) $8y_{n+1} = 280y_{n+3} - 48x_{n+3} - 1152$
10) $8y_{n+2} = 48y_{n+3} - 8x_{n+3} - 168$
11) $8x_{n+1} = 48y_{n+3} - 8x_{n+3} - 192$
12) $8x_{n+2} = 8y_{n+3} - 24$

1) Each of the following expressions is a nasty number

i) $48y_{2n+2} - 8x_{2n+2} - 192$

ii) $280y_{2n+3} - 48x_{2n+3} - 1176$

14)

$$\frac{1}{6}\left[48y_{3n+3} - 8x_{3n+3} - 204\right] + 3\left[48y_{n+1} - 8x_{n+1} - 204\right]$$

is a cubic number.

2.2. Remarkable Observations

1) By considering suitable linear transformations between the solutions of (1), one may get integer solutions for hyperbolas

Example 5) Define $X = 48y_{n+1} - 8x_{n+1} - 204$, $Y = -136y_{n+1} + 24x_{n+1} + 576$

Note that the pair (X, Y) satisfies the hyperbola $Y^2 = 8X^2 - 32 \times 6^2$

Example 6) Define

$$X = 1680 y_{n+2} - 288 x_{n+2} - 7128,$$

$$Y = -4752 y_{n+2} + 816 x_{n+2} + 20160$$

Note that the pair (X, Y) satisfies the hyperbola $Y^2 = 8X^2 - 32 \times 36^2$

2) By considering suitable linear transformations between the solutions of (1) , one may get integer solutions for parabolas

Example 7) Define $X = 48 y_{n+1} - 8 x_{n+1} - 204$, $Y = -136 y_{n+1} + 24 x_{n+1} + 576$

Note that the pair (X, Y) satisfies the parabola $Y^2 = 6 \times 8X - 32 \times 6^2$

Example 8) Define

$$X = 1680 y_{n+2} - 288 x_{n+2} - 7128,$$

$$Y = -4752 y_{n+2} + 816 x_{n+2} + 20160$$

Note that the pair (X, Y) satisfies the parabola $Y^2 = 36 \times 8X - 32 \times 36^2$

3. CONCLUSION

In this paper, we have made an attempt to obtain a complete set of non-trivial distinct solutions for the non-homogeneous binary quadratic equation. To conclude, one may search for other choices of solutions to the considered binary equation and further, quadratic equations with multi-variables.

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